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From normality to Q_m-normality ☆

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Abstract

The definition of a Q_m-normal family, $m \in \mathbb{N}$, is a geometrical extension of the notion of normality. In this paper we extend three conditions of normality and derive three other conditions for Q_m-normality.

Keywords: Q_m-normal family; Zalcman's Lemma; $(a)_m$-disk

1. Introduction

We require some notation and definitions before presenting the notion of Q_m-normality and introducing our results.

We denote $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0, r) = \{z \in \mathbb{C}\colon |z - z_0| < r\}$, $\Delta'(z_0, r) = \{z \in \mathbb{C}\colon 0 < |z - z_0| < r\}$, $\overline{\Delta}(z_0, r) = \{z \in \mathbb{C}\colon |z - z_0| \leqslant r\}$.

A disk K is said to be *compactly contained* in a domain D if $\overline{K} \subset D$. Two disks K_1, K_2 are *strongly disjoint* if $\overline{K}_1 \cap \overline{K}_2 = \emptyset$. If D is a domain and $E \subset D$, then the *derived set of* E with respect to D, denoted by $E_D^{(1)}$ is the set of accumulation points of E in D. For $k \geqslant 2$, the derived set of order k of E with respect to D is defined inductively by $E_D^{(k)} = (E_D^{(k-1)})_D^{(1)}$. Also $E_D^{(0)}$ is defined to be E.

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We write $f_n \overset{\chi}{\Rightarrow} f$ on D to indicate that the sequence $\{f_n\}$ of meromorphic functions on D converges to f uniformly on compact subsets of D in the spherical metric χ. We write $f_n \Rightarrow f$ on D if the convergence is already in the Euclidean metric.

The theory of Q_m-normal families was developed by X.T. Chuang [4]. Let $m \in \mathbb{N} \cup \{0\}$. A family $\mathcal{F}$ of functions, meromorphic on D is called Q_m-normal on D if each sequence $S = \{f_n\}$ of $\mathcal{F}$ has a subsequence $S' = \{f_{n_k}\}$ such that $f_{n_k} \overset{\chi}{\Rightarrow} f$ on $D \setminus E$, where f is a function on D (happens to be meromorphic or $f \equiv \infty$) and $E \subset D$ satisfies $E_D^{(m)} = \emptyset$. If $\nu \in \mathbb{N}$, then $\mathcal{F}$ is *Q_m-normal of order at most* ν on D if in addition S' can always be taken such that $|E_D^{(m-1)}| = \nu$.

We consider three (types of) normal families (on some domain D), all of which are defined by some condition, and each of which is extended to be a Q_m-normal family by letting each member of the family not satisfy the condition in some restricted set in D.

Definition 1.1. [4, Definition 8.13] Given a sequence of integers $I_1 = 1$, $I_m \geqslant 2$ $(m = 2, 3, \ldots)$ and a sequence of numbers $\eta_m > 0$ $(m = 1, 2, \ldots)$, consider a meromorphic function f in a domain D, an open disk Δ_0 of radius r and a value $a \in \widehat{\mathbb{C}}$. We say that Δ_0 is an $(a)_1$-disk of the function f in D if Δ_0 has the following properties:

(A_1) $\overline{\Delta}_0$ is contained in D.
(B_1) $r < \eta_1$.
(C_1) The function f takes the value a in Δ_0.

We say that Δ_0 is an $(a)_2$-disk of the function f in D if Δ_0 has property (A_1) as well as the following properties:

(B_2) $r < \eta_2$.
(C_2) The function f has at least I_2 strongly disjoint $(a)_1$-disks compactly contained in Δ_0.

In general, if $m \geqslant 2$ is an integer, we say that Δ_0 is an $(a)_m$-disk of the function f in D if Δ_0 has property (A_1) as well as the following properties:

(B_m) $r < \eta_m$.
(C_m) The function f has at least I_m strongly disjoint $(a)_{m-1}$-disks compactly contained in Δ_0.

Our three main results are:

Theorem 1.2. *Let $D \subset \mathbb{C}$ be a domain; $\nu, m, n \in \mathbb{N}$, $n \geqslant 3$; $a, b \in \mathbb{C}$, $a \neq 0$. Let $\mathcal{F} = \mathcal{F}_{m,\nu}(a, b, n)$ be the family of meromorphic functions on D such that $f' - af^n$ has at most ν strongly disjoint $(b)_m$-disks, compactly contained in D. Then $\mathcal{F}$ is a Q_m-normal family of order at most ν in D. Moreover, if the functions in $\mathcal{F}$ are holomorphic, the assertion is true for $n \geqslant 2$.*

Theorem 1.3. *Let $D \subset \mathbb{C}$ be a domain; $m, k \in \mathbb{N}$; $\nu_1, \nu_2 \geqslant 0$ integers. Let ψ be a meromorphic function on D, and $\mathcal{F}$ a family of meromorphic functions on D. Suppose that the following conditions hold*:

(1) $\psi \not\equiv 0$;
(2) *each $f \in \mathcal{F}$ has at most ν_1 strongly disjoint $(0)_m$-disks compactly contained in D*;
(3) *for each $f \in \mathcal{F}$, $f^{(k)} - \psi$ has at most ν_2 strongly disjoint $(0)_m$-disks compactly contained in D*;
(4) *no $f \in \mathcal{F}$ has poles in common with ψ in D.*

Then $\mathcal{F}$ is a Q_m-normal family of at most order $\nu_1 + \nu_2$ on D.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, and let a_1 and a_2 be distinct complex numbers. Assume that for each $f \in \mathcal{F}$ and each $j = 1, 2$ any $(a_j)_2$-disk of f in D is an $(a_j)_2$-disk of f' in D, and vice versa. Then $\mathcal{F}$ is a Q_1-normal family on D.*

The paper is organized as follows. Section 2 contains preliminary results. In Section 3, we give background on the normal version of Theorem 1.2 and give its proof. In Section 4, we do the same with relation to Theorem 1.3 and in Section 5 with relation to Theorem 1.4.

2. Preliminary results

First we give some definitions that lead us to alter definitions of Q_m-normal family. All definitions are taken from Chuang's book [4, Chapter 8].

Definition 2.1. (C_m-point, $m \geqslant 0$). Let $m \geqslant 0$ be an integer. Let $S = \{f_n\}$ be a sequence of functions meromorphic on a domain D. A point $z_0 \in D$ is called a C_0-point of S if for some $r > 0$ such that $\Delta(z_0, r) \subset D$, S converges uniformly in $\Delta(z_0, r)$ with respect to the spherical metric χ. For $m \geqslant 1$, a point $z_0 \in D$ is a C_m-point of S if for some $r > 0$ such that $\Delta(z_0, r) \subset D$ every point in $\Delta'(z_0, r)$ is a C_{m-1}-point of S. For $m \geqslant 0$, if z' is not a C_m-point of S then z' is called a non C_m-point of S.

Definition 2.2. (C_m-sequence, $m \geqslant 0$). Let $m \geqslant 0$ be an integer. A sequence S of functions meromorphic on a domain D is called a C_m-sequence in D if each $z \in D$ is a C_m-point of S.

Definition 2.3. (Q_m-normal family, $m \geqslant 0$). Let $m \geqslant 0$ be an integer. A family $\mathcal{F}$ of functions meromorphic in a domain D is called Q_m-normal in D, if each sequence $S = \{f_n\}$ of $\mathcal{F}$ has a subsequence $S' = \{f_{n_k}\}$ which is a C_m-sequence in D. If $m \geqslant 1$ and $\nu \geqslant 1$ are integers and S' can always be taken to have at most ν non C_{m-1}-points in D, then $\mathcal{F}$ is said to be Q_m-normal of order at most ν in D.

It turns out that a Q_0-normal family is just a normal family, and that a Q_1-normal family is a quasi-normal family. The latter notion was introduced by P. Montel [11] who developed also the classical notion of normality. In [4], these definitions were given successively in Chapters 1, 5 and 8 to Q_0-normality, Q_1-normality and to general Q_m-normality, respectively.

The equivalence of the definition that we first gave for Q_m-normal family and Definition 2.3 follows from the following lemma.

Lemma 2.4. [4, Theorem 8.2] *Let S be a sequence of meromorphic functions in a domain D, and $m \geqslant 0$ an integer. In order for S to be a C_m-sequence in D, it is necessary and sufficient that the set E of non C_0-points of S in D satisfies $E_D^{(m)} = \emptyset$.*

We shall also use the following lemmas.

Lemma 2.5. *If $E_1, E_2 \subset D$ and $m \geqslant 0$ is an integer, then $(E_1 \cup E_2)_D^{(m)} = (E_1)_D^{(m)} \cup (E_2)_D^{(m)}$.*

Lemma 2.6. *Let S be a sequence of meromorphic functions in a domain D, $z_0 \in D$, and $m \geqslant 0$ an integer. Then z_0 is a non C_m-point of S if and only if $z_0 \in E_D^{(m)}$, where E is the set of the non C_0-points of S in D.*

The proof of Lemma 2.6 follows at once by mathematical induction. Observe also that Lemma 2.4 is an immediate consequence of Lemma 2.6.

Lemma 2.7. *Let $m \geqslant 0$ be an integer, and let $E \subset D$. If $z_0 \in E_D^{(m)}$ and Δ_0 is a neighborhood of z_0 such that $\Delta_0 \subset D$, then $(\Delta_0 \cap E)_{\Delta_0}^{(m)} \neq \emptyset$ and $z_0 \in (\Delta_0 \cap E)_D^{(m)}$.*

The interested reader is referred also to [12] for a brief and concise background of Q_m-normality.

The following important lemma due to L. Zalcman is a very useful tool in the research of normal families.

Zalcman's Lemma. [20] *A family $\mathcal{F}$ of functions meromorphic* (*analytic*) *on the unit disk Δ is* not *normal if and only if there exist*

(a) *a number* $0 < r < 1$;
(b) *points* z_n, $|z_n| < r$;
(c) *functions* $f_n \in \mathcal{F}$; *and*
(d) *numbers* $\rho_n \to 0^+$,

such that

$$f_n(z_n + \rho_n \zeta) \overset{\chi}{\Rightarrow} g(\zeta) \quad \text{on } \mathbb{C},$$

where g is a nonconstant meromorphic (*entire*) *function on* $\mathbb{C}$.

Moreover, $g(\zeta)$ can be taken to satisfy the normalization $g^{\#}(\zeta) \leqslant g^{\#}(0) = 1$, $\zeta \in \mathbb{C}$.

For applications of Zalcman's Lemma, see [1,21,22].

In [12], Zalcman's Lemma was extended to be a criterion for non Q_m-normality or non Q_m-normality of order ν as in the following.

Lemma 2.8. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $m \geqslant 1$. In order that $\mathcal{F}$ not be a Q_m-normal family in D, it is necessary and sufficient that there exist*

(a) *a sequence of functions of $\mathcal{F}$, $S = \{f_n\}_{n=1}^{\infty}$;*
(b) *a set $E \subset D$ satisfying $E_D^{(m)} \neq \emptyset$, such that to each point $z \in E$ correspond;*
(c) *a sequence of points $\{\omega_{n,z}\}_{n=1}^{\infty}$ belonging to D such that $\omega_{n,z} \to z$;*
(d) *a sequence $\rho_{n,z} \to 0^+$; and*
(e) *a nonconstant function $g_z(\zeta)$, meromorphic on $\mathbb{C}$, such that*
(f) *$f_n(\omega_{n,z} + \rho_{n,z}\zeta) \stackrel{\chi}{\Rightarrow} g_z(\zeta)$ on $\mathbb{C}$.*

Lemma 2.9. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $m, \nu \geqslant 1$ integers. In order that $\mathcal{F}$ not be a Q_m-normal family of order at most ν in D, it is necessary and sufficient that there exist*

(a) *a sequence of functions of $\mathcal{F}$, $S = \{f_n\}_{n=1}^{\infty}$;*
(b) *a set $E \subset D$ satisfying $|E_D^{(m-1)}| \geqslant \nu + 1$, such that to each point $z \in E$ correspond;*
(c) *a sequence of points $\{\omega_{n,z}\}_{n=1}^{\infty}$ belonging to D such that $\omega_{n,z} \to z$;*
(d) *a sequence $\rho_{n,z} \to 0^+$; and*
(e) *a nonconstant function $g_z(\zeta)$, meromorphic on $\mathbb{C}$, such that*
(f) *$f_n(\omega_{n,z} + \rho_{n,z}\zeta) \stackrel{\chi}{\Rightarrow} g_z(\zeta)$ on $\mathbb{C}$.*

3. Theorem 1.2 and its background

Let $a, b \in \mathbb{C}$, $a \neq 0$ and $n \in \mathbb{N}$. Let $\mathcal{F} = \mathcal{F}(a, b, n)$ be the family of meromorphic functions defined on Δ by the rule that $f \in \mathcal{F}$ if and only if $f' - af^n \neq b$ on Δ. Over the past thirty years, the question of the normality of $\mathcal{F}$ (depending on the value of n and on whether the functions of $\mathcal{F}$ are meromorphic or holomorphic) has been thoroughly investigated. The starting point was Hayman's paper [6], where he proved that a meromorphic function on $\mathbb{C}$ which satisfies $f' - af^n \neq b$ must be constant if $n \geqslant 5$; when f is entire, $n \geqslant 3$ suffices. For analytic functions, the normality result corresponding to Hayman's theorem was proved by Drasin [5]. The corresponding result for meromorphic functions was established (independently) by Langley [7], Song-Ying Li [8], and Xianjin Li [10]; cf. Li and Xie [9]. More recently, based on the Zalcman–Pang Lemma, Pang [14] proved that the condition $f' - af^4 \neq b$ implies normality for families of meromorphic functions. The sufficiency of the condition $f' - af^2 \neq b$ for normality of families of analytic functions follows in a similar way, cf. [19]. Finally, from the (independently obtained) results of Chen and Fang [3], Bergweiler and Eremenko [2], and Zalcman [21], it follows that the family of meromorphic functions which satisfies $f' - af^3 \neq b$ is a normal family.

To summarize, we can formulate

Theorem 3.1. *The family $\mathcal{F} = \mathcal{F}(a, b, n)$ is normal on Δ when the functions of $\mathcal{F}$ are meromorphic and $n \geqslant 3$ or in case $n \geqslant 2$ and the functions are holomorphic.*

Since normality is a local property, Theorem 3.1 remains valid if Δ replaced by an arbitrary domain, $D \subseteq \mathbb{C}$.

By Theorem 3.1, we now give the proof of Theorem 1.2 and some related results.

Proof of Theorem 1.2. The proof is the same for meromorphic or holomorphic functions and depends only on Theorem 3.1. Accordingly, we treat only the case of meromorphic functions, $n \geqslant 3$. We proceed by induction on m.

$m = 1$. Let $\{f_k\}$ be a sequence of $\mathcal{F}$. By Definition 1.1 for $m = 1$, there exist for each $k \geqslant 1$ points $z_{k,1}, \ldots, z_{k,\nu_k}$ of D, $0 \leqslant \nu_k \leqslant \nu$, such that if $z \in D \setminus \{z_{k,1}, \ldots, z_{k,\nu_k}\}$ then

$$f_k'(z) - af_k^n(z) \neq b \tag{1}$$

(if $\nu_k = 0$ then (1) is true for all $z \in D$). By passing to a subsequence if necessary, we can assume that $\nu_k = \nu_0$, $k \geqslant 1$; $0 \leqslant \nu_0 \leqslant \nu$ and that

$$z_{i,k} \underset{k\to\infty}{\to} z_i, \quad 1 \leqslant i \leqslant \nu_0 \quad \text{where } z_i \in \overline{D} \subset \widehat{\mathbb{C}}. \tag{2}$$

Set $D_0 = D \setminus \{z_1, \ldots, z_{\nu_0}\}$ (if $\nu_0 = 0$ then $D_0 = D$) and take $z_0 \in D_0$. Take $r > 0$ such that $\overline{\Delta}(z_0, r) \subset D_0$; then by (1) and (2), it follows that for large enough k, $f_k' - af_k^n \neq b$ on $\Delta(z_0, r)$. Hence by Theorem 3.1, $\{f_k\}$ is normal in D_0. Therefore, $\{f_k\}$ has a subsequence $\{f_{k_\ell}\}$ which is a C_0-sequence in D_0; and it is clear that $\{f_{k_\ell}\}$ has at most ν non C_0-points in D. By Definition 2.3, $\mathcal{F}$ is a Q_1-normal family of order at most ν in D.

Assume the correctness of the theorem for m and consider the family $\mathcal{F} = \mathcal{F}_{m+1,\nu}(a, b, n)$. Suppose that $\mathcal{F}$ is not a Q_{m+1}-normal family of order at most ν in D. By Lemma 2.9, we have a set $E \subset D$ satisfying $|E_D^{(m)}| \geqslant \nu + 1$ and for every $z \in E$ corresponding sequences $\{f_k\}$, $\{\rho_{k,z}\}$, and $\{\omega_{k,z}\}$ as in Lemma 2.9 such that

$$f_k(\omega_{k,z} + \rho_{k,z}\zeta) \overset{\chi}{\Rightarrow} g_z(\zeta) \quad \text{on } \mathbb{C}, \tag{3}$$

where g_z is a nonconstant meromorphic function on $\mathbb{C}$.

Let $b_1, b_2, \ldots, b_{\nu+1}$ be distinct points in $E_D^{(m)}$. Construct $\nu + 1$ strongly disjoint disks compactly contained in D, $\Delta_i = \Delta(b_i, r_i)$, $r_i < \eta_{m+1}$, $1 \leqslant i \leqslant \nu + 1$ (see Definition 1.1 for η_{m+1}). We claim that for each $1 \leqslant i \leqslant \nu + 1$, Δ_i is a $(b)_{m+1}$-disk of $f_k' - af_k^n$ for large enough k. Indeed, suppose this is not true. Then there exists a subsequence $\{f_{k_\ell}\}_{\ell=1}^\infty$ ($k_\ell = k_\ell(i)$) such that for each $\ell \geqslant 1$, Δ_i is not a $(b)_{m+1}$-disk of $f_{k_\ell}' - af_{k_\ell}^n$. By Definition 1.1, there are at most $I_{m+1} - 1$ strongly disjoint $(b)_m$-disks of $f_{k_\ell}' - af_{k_\ell}^n$ compactly contained in Δ_i. By the assumption for m, $\{f_{k_\ell}\}$ is Q_m-normal of at most order $I_{m+1} - 1$ in Δ_i. In particular, $\{f_{k_\ell}\}$ is a Q_m-normal family there. On the other hand, by Lemma 2.7, $(E \cap \Delta_i)_{\Delta_i}^{(m)} \neq \emptyset$; and by (3), for each $z \in E \cap \Delta_i$

$$f_{k_\ell}(\omega_{k_\ell,z} + \rho_{k_\ell,z}\zeta) \overset{\chi}{\Rightarrow} g_z(\zeta) \quad \text{on } \mathbb{C}.$$

Applying Lemma 2.8, we see that $\{f_{k_\ell}\}$ is not a Q_m-normal family on Δ_i, a contradiction. Therefore, there is some $k_0 \geqslant 1$ such that for any $k \geqslant k_0$, Δ_i is a $(b)_{m+1}$-disk of $f_k' - af_k^n$. Thus $\Delta_1, \ldots, \Delta_{\nu+1}$ are $\nu+1$ strongly disjoint $(b)_{m+1}$-disks of $f_k' - af_k^n$ compactly contained in D, and we get a contradiction to the assumption for $m+1$. This completes the proof. □

Remark. Compare this proof to the proof of Lemma 1.52 [4, pp. 333–334], which of course makes no use of the extensions of Zalcman's Lemma.

From Theorem 1.2 we deduce

Theorem 3.2. *Let $D \subset \mathbb{C}$ be a domain, $m \geqslant 0$, $n \geqslant 3$ integers*; *$a, b \in \mathbb{C}$, $a \neq 0$. Let $\mathcal{F} = \mathcal{F}_m(a, b, n)$ be the family of meromorphic functions on D such that $f' - af^n$ has no $(b)_{m+1}$-disk in D. Then $\mathcal{F}$ is a Q_m-normal family on D. Moreover, if the functions are holomorphic, the assertion holds for $n \geqslant 2$.*

Proof. For $m = 0$, the result follows from Theorem 3.1 by Definition 1.1. Let $m \geqslant 1$ and suppose, to the contrary, that the family $\mathcal{F} = \mathcal{F}_m(a, b, n)$ is not a Q_m-normal family on D. According to Lemma 2.8, we have

$$f_k(\omega_{k,z} + \rho_{k,z}\zeta) \Rightarrow g_z(\zeta) \quad \text{on } \mathbb{C},\ z \in E, \tag{4}$$

where $E \subset D$ satisfies $E_D^{(m)} \neq \emptyset$, and $\{f_k\}$, $\{\rho_{k,z}\}$, $\{\omega_{k,z}\}$ and g_z are as in Lemma 2.8. Take $z_0 \in E_D^{(m)}$ and construct a disk $\Delta(z_0, r)$ such that $r < \eta_{m+1}$ (see Definition 1.1) and $\overline{\Delta}(z_0, r) \subset D$. By assumption, if $f \in \mathcal{F}$, $f' - af^n$ has at most $I_{m+1} - 1$ strongly disjoint $(b)_m$-disks compactly contained in $\Delta(z_0, r)$. Then, by Theorem 1.2, $\mathcal{F}$ is a Q_m-normal family on $\Delta(z_0, r)$ of at most order $I_{m+1} - 1$; in particular, $\mathcal{F}$ is a Q_m-normal family there. On the other hand, by Lemma 2.7, $(E \cap \Delta(z_0, r))_{\Delta(z_0,r)}^{(m)} \neq \emptyset$; and since (4) holds for any $z \in E \cap \Delta(z_0, r)$, we get by Lemma 2.8 that $\mathcal{F}$ is not a Q_m-normal family on $\Delta(z_0, r)$, a contradiction. □

Remark. Theorems 1.2 and 3.2 can be generalized in the same fashion when the condition $f' - af^n \neq b$ is replaced in any condition of the form $P(f, f', \ldots, f^{(k)}) \neq a$ that insures normality, where P is some polynomial in $k+1$ variables, $k \geqslant 0$, $a \in \mathbb{C}$.

4. Background to Theorem 1.3 and its proof

The following theorem was first proved by Yang Le [18] (for ψ analytic) and later generalized by W. Schwick [16] to meromorphic ψ. In [13] a somewhat different proof is given. The proof of Theorem 1.3 is based on this theorem.

Theorem 4.1. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let k be a positive integer. Suppose that ψ is a meromorphic function in D, and the following conditions hold*:

(1) $\psi \not\equiv 0$;
(2) $f(z) \neq 0$, $z \in D$, $f \in \mathcal{F}$;
(3) $f^{(k)}(z) \neq \psi(z)$, $z \in D$, $f \in \mathcal{F}$;
(4) *no* $f \in \mathcal{F}$ *has poles in common with* ψ *in* D.

Then $\mathcal{F}$ *is a normal family on* D.

Proof of Theorem 1.3. Induction on m. (For convenience, we write $\mathcal{F} = \mathcal{F}_m$ at the mth step of the induction.)

$m = 1$.

Let $\{f_n\}$ be a sequence of $\mathcal{F} = \mathcal{F}_1$.

As in the proof of Theorem 1.2, we can assume without loss of generality that for each $n \geqslant 1$, f_n has N_1 distinct roots in D, $\zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,N_1}$, $0 \leqslant N_1 \leqslant \nu_1$; that $f_n^{(k)} - \psi$ has N_2 distinct roots in D, $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,N_2}$, $0 \leqslant N_2 \leqslant \nu_2$; and that, for each $1 \leqslant i \leqslant N_1$, $1 \leqslant j \leqslant N_2$,

$$\zeta_{n,i} \underset{n\to\infty}{\to} \zeta_i \in \overline{D}, \qquad \xi_{n,j} \underset{n\to\infty}{\to} \xi_j \in \overline{D}.$$

Consider the domain

$$D_0 = D \setminus \{\zeta_1, \ldots, \zeta_{N_1}, \xi_1, \ldots, \xi_{N_2}\}.$$

It follows from Theorem 4.1 that $\{f_n\}$ is normal in D_0. Hence it has a subsequence $\{f_{n_\ell}\}$ with at most $N_1 + N_2$ non C_0-points in D, $N_1 + N_2 \leqslant \nu_1 + \nu_2$; and so we are done. In the case $N_1 = N_2 = 0$, $\{f_{n_\ell}\}$ is a C_0-sequence in D (cf. the proof for $m = 1$ in Theorem 1.2).

Assume now that the theorem is correct for m and let us prove it for $m + 1$. Suppose, to the contrary, that $\mathcal{F} = \mathcal{F}_{m+1}$ is not a Q_{m+1}-normal family of order at most $\nu_1 + \nu_2$ in D. By Lemma 2.9, we have

$$f_n(\omega_{n,z} + \rho_{n,z}\zeta) \overset{\chi}{\Rightarrow} g_z(\zeta) \quad \text{on } \mathbb{C}, \tag{5}$$

for each $z \in E$, where $E \subset D$ satisfies $|E_D^{(m)}| \geqslant \nu_1 + \nu_2 + 1$ and $\{f_n\}$, $\{\rho_{n,z}\}$, $\{\omega_{n,z}\}$ are as in Lemma 2.9.

Let $b_1, b_2, \ldots, b_{\nu_1+\nu_2+1}$ be $\nu_1 + \nu_2 + 1$ distinct points of $E_D^{(m)}$. Construct $\nu_1 + \nu_2 + 1$ strongly disjoint disks compactly contained in D, $\Delta_i = \Delta(b_i, r_i)$, $r_i < \eta_{m+1}$, $1 \leqslant i \leqslant \nu_1 + \nu_2 + 1$. We assert that, for each $1 \leqslant i \leqslant \nu_1 + \nu_2 + 1$ and sufficiently large n, Δ_i is either an $(0)_{m+1}$-disk of f_n or an $(0)_{m+1}$-disk of $f_n^{(k)} - \psi$, which is a contradiction to the assumption for $m + 1$ and therefore proves the theorem. Indeed, suppose that this is not the case. Then $\{f_n\}$ has a subsequence $\{f_{n_\ell}\}$ such that for each $\ell \geqslant 1$, Δ_i is neither an $(0)_{m+1}$-disk of f_{n_ℓ} nor an $(0)_{m+1}$-disk of $f_{n_\ell}^{(k)} - \psi$. By Definition 1.1, there are at most $I_{m+1} - 1$ strongly disjoint $(0)_m$-disks of f_{n_ℓ} compactly contained in Δ_i and at most $I_{m+1} - 1$ strongly disjoint $(0)_m$-disks of $f_{n_\ell}^{(k)} - \psi$ compactly contained in Δ_i. By the induction assumption, $\{f_{n_\ell}\}$ is Q_m-normal of order at most $2I_{m+1} - 2$ in Δ_i. We then get a contradiction to (5) via Lemma 2.7 (cf. Theorem 1.2). $\square$

In the same fashion that Theorem 3.2 was derived from Theorem 1.2, we can derive from Theorem 1.3 the following result.

Theorem 4.2. *Let $D \subset \mathbb{C}$ be a domain, $m, k \in \mathbb{N}$. Let ψ be a meromorphic function on D, and $\mathcal{F}$ ($\mathcal{F} = \mathcal{F}_m$) a family of meromorphic functions on D. Suppose that the following conditions hold*:

(1) $\psi \not\equiv 0$;
(2) *no $f \in \mathcal{F}$ has an $(0)_{m+1}$-disk in D*;
(3) *for each $f \in \mathcal{F}$, $f^{(k)} - \psi$ has no $(0)_{m+1}$-disk in D*;
(4) *no $f \in \mathcal{F}$ has poles in common with ψ in D.*

Then $\mathcal{F}$ is a Q_m-normal family on D.

Remark. Assume we omit condition (4) in Theorem 4.2, and let $A = \{z_1, z_2, \ldots\}$ be the set of poles of ψ in D. Evidently, the family $\mathcal{F} = \mathcal{F}_m$ from Theorem 4.2 satisfies conditions (1)–(4) in the domain $D_0 = D \setminus A$; and, by the theorem, $\mathcal{F}$ is a Q_m-normal family on D_0. Let S be a sequence of functions of $\mathcal{F}$; then by Definition 2.3, it has a subsequence S^* which is a C_m-sequence in D_0.

Denote by E_0 the set of non C_0-points of S^* in D_0; by Lemma 2.6, $(E_0)_{D_0}^{(m)} = \emptyset$. Now, if E is the set of non C_0-points of S^* in D, then $E \subset E_0 \cup A$; and by Lemma 2.5, $E_D^{(m+1)} \subset (E_0)_D^{(m+1)} \cup A_D^{(m+1)}$. Since A has no accumulation points in D, $A_D^{(m+1)} = \emptyset$. Likewise, by Lemma 2.7, if $z_0 \in (E_0)_D^{(m)}$, then $z_0 \in A$ (else $z_0 \in (E_0)_{D_0}^{(m)} = \emptyset$), and so $E_D^{(m+1)} \subset (E_0)_D^{(m+1)} = ((E_0)_D^{(m)})_D^{(1)} \subset A_D^{(1)} = \emptyset$; hence $E_D^{(m+1)} = \emptyset$. Thus, again by Lemma 2.6, S^* is a C_{m+1}-sequence in D, and by Definition 2.3, $\mathcal{F}$ is a Q_{m+1}-normal family on D. In particular, if A is finite, we get that $\mathcal{F}$ is Q_{m+1}-normal of order at most $|A|$ in D.

In a similar fashion, if we omit condition (4) from the assumptions of Theorem 1.3 and define A and D_0 in the same way, then any sequence $S \subset \mathcal{F}$ has a subsequence S^* with at most $\nu_1 + \nu_2$ non C_{m-1}-points in D_0. Since a non C_m-point is an accumulation point of non C_{m-1}-points (see Definition 2.1), no pole of ψ can be a non C_m-point of S^* in D (because these poles are isolated in D, and D_0 contains only a finite number of non C_{m-1}-points of S^*). Therefore, S^* is a Q_m-normal family on D. In particular, if A is finite, then $\mathcal{F}$ is a Q_m-normal family of order at most $\nu_1 + \nu_2 + |A|$ in D.

5. Sharing values: Theorem 1.4 and its proof

Definition. Let f and g be meromorphic functions in the domain D, and let $a \in \mathbb{C}$. Then f and g are said to *share the value a in D*, if $g^{-1}(\{a\}) \cap D = f^{-1}(\{a\}) \cap D$.

Schwick seems to have been the first to draw a connection between normality criteria and shared values. He proved [17]

Theorem 5.1. *Let $\mathcal{F}$ be a family of meromorphic functions on the domain D and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3 for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Later, Pang and Zalcman [15] improved this result. They proved

Theorem 5.2. *Let $\mathcal{F}$ be a family of meromorphic functions on the domain D, and let a and b be distinct complex numbers. If f and f' share a and b for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

We now prove

Theorem 5.3. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $\nu \geqslant 1$ an integer, and a and b distinct complex numbers. Assume that for each $f \in \mathcal{F}$ there exist points $z_1^{(f)}, z_2^{(f)}, \ldots, z_{n_f}^{(f)}$, $n_f \leqslant \nu$, belonging to D such that f and f' share a and b in $D \setminus \{z_1^{(f)}, z_2^{(f)}, \ldots, z_{n_f}^{(f)}\}$. Then $\mathcal{F}$ is a Q_1-normal family of order at most ν in D.*

Proof. Let $S = \{f_k\}$ be a sequence in $\mathcal{F}$. By passing to a subsequence, if necessary, we can assume, without loss of generality, that for each k, $n_{f_k} = N$, $0 \leqslant N \leqslant \nu$. This means that for each $k \geqslant 1$, f_k and f'_k share a and b in the domain $D \setminus \{\zeta_1^{(k)}, \zeta_2^{(k)}, \ldots, \zeta_N^{(k)}\}$ (or in D if $N = 0$, in which case $\{f_k\}$ is normal according to Theorem 5.2). We can also assume that for each $1 \leqslant i \leqslant N$, $\zeta_i^{(k)} \underset{k\to\infty}{\to} \zeta_i \in \overline{D}$.

Take a point $\zeta_0 \in D_0 = D \setminus \{\zeta_1, \ldots, \zeta_N\}$ and let $r > 0$ be such that $\overline{\Delta}(\zeta_0, r) \subset D_0$. For k large enough, say $k \geqslant k_0$, $\zeta_i^{(k)} \notin \Delta(\zeta_0, r)$ for each $1 \leqslant i \leqslant N$. Hence by Theorem 5.2, S is normal in $\Delta(\zeta_0, r)$. Since normality is a local property, we deduce that S is normal in D_0, so it has a subsequence S' which is a C_0-sequence in D_0. Evidently, S' has at most ν non C_0-points in D; so by Definition 2.3, $\mathcal{F}$ is a Q_1-normal family of order at most ν in D. $\square$

We now are almost in a position to prove Theorem 1.4. But first we need the following lemma.

Lemma 5.4. *Let $\Delta(\zeta_0, \rho)$ be a disk in $\mathbb{C}$, and let $n \geqslant 2$ be an integer. Suppose we are given three collections of points designated as follows*:

Points colored white will be called W-points;
points colored black will be called B-points;
points colored both colors, black and white will be called BW-points.

Suppose the following conditions hold:

(1) *$\Delta(\zeta_0, \rho)$ contains a finite number and at least n W-points*;
(2) *$\Delta(\zeta_0, \rho)$ contains a finite number of B-points*;
(3) *$\Delta(\zeta_0, \rho)$ contains a finite number of BW-points.*

Then there exists an open disk Δ^*, compactly contained in $\Delta(\zeta_0, \rho)$, which satisfies one of the following:

(I) *Δ^* contains at least n points with W and at most $n-1$ points with B;*
(II) *Δ^* contains at least n points with B and at most $n-1$ points with W.*

Here "point with W (B)" means a point that is either a W-point (B-point) or a BW-point.

Proof of Lemma 5.4. Without loss of generality, take $\Delta(\zeta_0, \rho)$ to be the unit disk Δ. Let z_0 be a W-point such that $|z_0|$ is maximal (possibly, there is more than one such point). Without loss of generality, assume that $z_0 = iy_0$, $y_0 > 0$. By the property of z_0 and condition (1), $\overline{\Delta}(0, y_0)$ contains at least n points with W. Define

$$r = \sup\big\{R \geqslant 0\colon\ \overline{\Delta}(iR, y_0 - R) \text{ contains at least } n \text{ points with } W\big\}.$$

We have $r \geqslant 0$; and since the number of points with W is finite (by conditions (1) and (3)), r is actually a maximum satisfying $0 \leqslant r < y_0$. Hence the open disk $\Delta(ir, y_0 - r)$ contains fewer than n points with W. Note that $z_0 \in \Gamma = \partial\Delta(ir, y_0 - r)$. Now, if $\Delta(ir, y_0 - r)$ contains at least n points with B, set $\Delta^* = \Delta(ir, y_0 - r)$, and condition (II) holds. Therefore, assume that $\Delta(ir, y_0 - r)$ contains $n-k$ points with B, $1 \leqslant k \leqslant n$ and $n - \ell$ points with W, $1 \leqslant \ell \leqslant n$. Let $0 < r_1 < y_0 - r$ be such that all the colored points in $\Delta(ir, y_0 - r)$ are already contained in $\overline{\Delta}(ir, r_1)$.

As a set, Γ contains at least ℓ points with W, one of which is z_0. Now let $\Gamma_1 \subset \Gamma$ be an arc which contains exactly ℓ points with W and has z_0 as one of its endpoints and some point with W as its second endpoint. (If $\ell \geqslant 2$, there exist exactly two such subarcs of Γ; if $\ell = 1$, then $\Gamma_1 = \{z_0\}$).

If Γ_1 contains fewer than k points with B, construct an open disk Δ_W compactly contained in Δ which contains $\Gamma_1 \cup \overline{\Delta}(ir, r_1)$ and contains no colored points except those which lie on $\Gamma_1 \cup \overline{\Delta}(ir, r_1)$. The existence of such a disk is guaranteed by the property of r_1 and the fact that there are a finite number of colored points in Δ. Certainly, Δ_W satisfies condition (I).

On the other hand, if Γ_1 contains at least k points with B, we cut from Γ_1 a small enough piece that contains only z_0 from all the colored points which lie on Γ_1 and create an arc $\Gamma_2 \subset \Gamma_1$ that contains $\ell - 1$ points with W and at least k points with B. In a fashion similar to the construction of Δ_W, construct an open disk Δ_B, compactly contained in Δ, which contains $\Gamma_2 \cup \overline{\Delta}(ir, r_1)$, and no colored points except those of this union. The disk Δ_B evidently satisfies condition (II). This completes the proof of the lemma. □

Proof of Theorem 1.4. Suppose, to the contrary, that $\mathcal{F}$ is not Q_1-normal. Using Lemma 2.8, we get a sequence $S = \{f_k\}$ of $\mathcal{F}$ and a sequence of distinct points in D, $z_n \to \zeta_0 \in D$, such that

$$\text{for every } n \geqslant 1,\ z_n \text{ is a non } C_0\text{-point for each subsequence of } S. \tag{6}$$

Take ρ,

$$0 < \rho < \eta_2 \tag{7}$$

(see Definition 1.1 for η_2), such that

$$\overline{\Delta}(\zeta_0, \rho) \subset D. \tag{8}$$

In view of (8), there are two alternatives.

(I). For infinitely many values of k, one of the following four possibilities holds.

$$\text{For } j=1 \text{ or } j=2, \quad f_k(z)\equiv a_j \text{ in } D. \tag{9}$$

$$\text{For } j=1 \text{ or } j=2, \quad f_k(z)=a_j z+b_k \text{ in } D. \tag{10}$$

So, in any case, there exists a subsequence $S'=\{f_{k_\ell}\}$ of S satisfying (9) or (10). If, say, $f_{k_\ell}\equiv a_1$, $\ell\geqslant 1$, then S' is a C_0-sequence in D, contradicting (6). If, again without loss of generality, $f_{k_\ell}(z)=a_1 z+b_{k_\ell}$, $\ell\geqslant 1$, then by Marty's Theorem, S' is a C_0-sequence in D (since $f^{\#}_{k_\ell}(z)\leqslant |a_1|,\ z\in D$), which again contradicts (6).

(II). There exists $k_0\geqslant 1$, such that for each $k\geqslant k_0$ and $j=1,2$, f_k and f'_k assume the value a_j only finitely often in $\Delta(\zeta_0,\rho)$. For some $n_0\geqslant 1$, $z_n\in\Delta(\zeta_0,\rho)$ for $n\geqslant n_0$. Construct $4I_2-3$ disks,

$$\Delta_i=\Delta(z_{n_0+i},r_i), \quad 1\leqslant i\leqslant 4I_2-3 \tag{11}$$

such that $\Delta_1,\Delta_2,\dots,\Delta_{4I_2-3}$ are strongly disjoint and compactly contained in $\Delta(\zeta_0,\rho)$. By (6), for each $1\leqslant i\leqslant 4I_2-3$, there exists $k_i\geqslant 1$ such that for each $k\geqslant k_i$ there exists $\zeta_i^{(k)}\in\Delta_i$ with

$$f_k(\zeta_i^{(k)})=a_j, \quad f'_k(\zeta_i^{(k)})\neq a_j \text{ (or vice versa)} \quad \text{for } j=1 \text{ or } j=2. \tag{12}$$

(Otherwise, there exists a subsequence of S $\{f_{k_\ell}\}$, such that for each $\ell\geqslant 1$, f_{k_ℓ} and f'_{k_ℓ} share a_1 and a_2 in Δ_i; then by Theorem 5.2, $\{f_{k_\ell}\}$ as a family is normal in Δ_i, violating (6).) Now take $N_1\geqslant\max\{k_1,\dots,k_{4I_2-3},k_0\}$. Because the disks in (11) are strongly disjoint, one of the four options in (12) must take place at (at least) I_2 distinct points of the system $\{\zeta_1^{(N_1)},\zeta_2^{(N_1)},\dots,\zeta_{4I_2-3}^{(N_1)}\}$ contained in $\Delta(\zeta_0,\rho)$. Without loss of generality, we can assume that

$$f_{N_1}(\zeta_i^{(N_1)})=a_1, \quad f'_{N_1}(\zeta_i^{(N_1)})\neq a_1 \quad \text{for } 1\leqslant i\leqslant I_2.$$

Now, by assumption there are a finite number of points, $z\in\Delta(\zeta_0,\rho)$, for which

$$f_{N_1}(z)=a_1, \qquad f'_{N_1}(z)\neq a_1. \tag{13}$$

Denote this number by N; of course,

$$N\geqslant I_2. \tag{14}$$

Now take $z\in\Delta(\zeta_0,\rho)$. If z satisfies (13), we say that z is a W-point. If z satisfies $f_{N_1}(z)\neq a_1$, $f'_{N_1}(z)=a_1$, we say that z is a B-point. If z satisfies $f_{N_1}(z)=f'_{N_1}(z)=a_1$, we say that z is a BW-point.

By the assumption and (14), all the conditions of Lemma 5.4 are fulfilled (with $n=I_2$). Therefore, its conclusion must hold, too. Assume, without loss of generality, that possibility (I) in Lemma 5.4 occurs. Then there exists an open disk Δ^*, compactly contained in $\Delta(\zeta_0,\rho)$, with at least I_2 points with W, and less than I_2 points with B. Translating the meaning of colors, together with (7) and Definition 1.1 for $m=1,2$, we see that $\Delta^*\subset D$ is an $(a_1)_2$-disk of f_{N_1}, but not an $(a_1)_2$-disk of f'_{N_1}. This contradicts the assumption of the theorem, and so the proof is complete. □

With the aid of Lemma 2.9 for $m = 2$, we can derive the following result from Theorem 1.4.

Theorem 5.5. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, and let a_1 and a_2 be distinct complex numbers. Assume that for each $f \in \mathcal{F}$, the total number of strongly disjoint disks compactly contained in D (in any configuration), each of which is an $(a_i)_2$-disk of f but not an $(a_i)_2$-disk of f', or vice versa, for $i = 1$ or $i = 2$, does not exceed ν. Then $\mathcal{F}$ is a Q_2-normal family of order at most ν on D.*

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